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ON PERIODIC SHADOWING OF INDUCED HYPERSPACE MAPS

Namjip Koo*, Hyunhee Lee**, and Nyamdavaa Tsegmid***

Abstract. In this paper we deal with the preservation of the periodic shadowing property of induced hyperspatial systems. More precisely, we show that an expansive system has the periodic shadowing property if and only if its induced hyperspatial system has the periodic shadowing property.

1. Introduction and preliminaries

In the theory of dynamical systems the definition of shadowing has several variations and directions. Fernándes and Good [4] studied the properties of shadowing and h-shadowing for the induced maps on hyperspaces via a general result on shadowing properties in dense subspaces. Also, Good *et al.* [5] studied preservation of various shadowing properties in discrete dynamical systems under inverse limits, products, factor maps and the induced maps for symmetric products and hyperspaces.

Kościelniak and Mazur introduced the notion of the periodic shadowing property for homeomorphisms on a compact metric space and showed that the periodic shadowing property is C^0 generic in the space of discrete dynamical systems on a compact smooth manifold M(see [7, 8]). Méndez [10] studied the connection concerning the density of two periodic sets $\mathrm{Per}(f)$ in X and $\mathrm{Per}(2^f)$ in 2^X. Darabi and Forouzanfar [3] showed that expansive dynamical systems with the shadowing property have the periodic shadowing property. Also, they showed that transitive dynamical systems with the periodic shadowing property has the shadowing property. Koo and Tsegmid [6] showed that if $f|_A : A \to A$ has the periodic shadowing property on an f-invariant dense subspace

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Correspondence should be addressed to Namjip Koo, njkoo@cnu.ac.kr.

A of X, then a continuous map $f : X \to X$ has the periodic shadowing property.

A dynamical system is a pair (X, f) consisting of a compact metric space X and a homeomorphism $f : X \to X$. Consider the following hyperspaces of a compact metric space X with a metric d and the induced maps on hyperspaces by a homeomorphism $f : X \to X$:

- $2^X = \{A \subseteq X \mid A$ is nonempty and closed$\}$ is the *hyperspace of closed nonempty subsets* of X;
- $\mathcal{F}_n(X) = \{A \in 2^X \mid A$ has at most n points$\}$ is the *n-fold symmetric product of* X;
- $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ is the collection of all finite subsets of X.
- $2^f : 2^X \to 2^X$ is given by $2^f(A) = f(A)$;
- $f^{<\omega} : \mathcal{F}(X) \to \mathcal{F}(X)$ is given by $f^{<\omega} = 2^f|_{\mathcal{F}(X)}$.

All hyperspaces mentioned above are considered with the Hausdorff metric d_H which is defined by

$$d_H(C, D) = \inf\{\epsilon > 0 \mid C \subseteq B_d(D, \epsilon) \text{ and } D \subseteq B_d(C, \epsilon)\}$$

for each pair of $C, D \in 2^X$. Here $B_d(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}$.

Now we recall the definitions of some dynamical properties that are used in this paper.

Let $\delta > 0$. We say that a bi-infinite sequence $\{x_n\}_{n\in\mathbb{Z}}$ of X is a *δ-pseudo orbit* of f if $d(f(x_n), x_{n+1}) \leq \delta$ for all $n \in \mathbb{Z}$. Given $\epsilon > 0$ we say that $\{x_n\}_{n\in\mathbb{Z}}$ can be *ϵ-shadowed* if there is $x \in X$ such that $d(f^n(x), x_n) \leq \epsilon$ for all $n \in \mathbb{Z}$. The dynamical system (X, f) is said to have the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ϵ-shadowed. We say that the dynamical system (X, f) has the *periodic shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$(i.e., $x_i = x_{i+k}$ for some $k \in \mathbb{N}$) can be ϵ-shadowed by some periodic point of f (see [8]). We say that the dynamical system (X, f) has the *strong periodic shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ with period N can be ϵ-shadowed by some periodic point of f with period N(see [2]). We say that the dynamical system (X, f) is *chain trasitive* if for any $x, y \in X$ and $\delta > 0$ there exists a δ-chain from x to y(i.e., a finite δ-pseudo-orbit of f). We say that the dynamical system (X, f) is *expansive* if there exists a constant $\delta > 0$ such that $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ implies $x = y$(see [12]). This means that $\Gamma_\delta^f(x) = \{x\}$ for each $x \in X$, where $\Gamma_\delta^f(x)$ is the *dynamical δ-ball* of f centered at x as follows

$$\Gamma_\delta^f(x) = \{y \in X \mid d(f^n(x), f^n(y)) \leq \delta,\ \forall n \in \mathbb{Z}\}.$$

In this paper we show that if a dynamical system (X, f) has the periodic shadowing property, then its induced hyperspatial system $(2^X, 2^f)$ has the periodic shadowing property. Also, we show that the converse to the above result holds under some conditions of systems (X, f) and $(2^X, 2^f)$. More precisely, we state our main result.

THEOREM 1.1. *An expansive system (X, f) has the periodic shadowing property if and only if the corresponding system $(2^X, 2^f)$ has the periodic shadowing property.*

2. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1 and establish the relation on the density of periodic points in homeomorphisms and their induced maps. For proof of Theorem 1.1, we need the following result which is adapted from Theorem 3.3 in [4] for the periodic shadowing property of homeomorphisms on a compact metric space.

LEMMA 2.1. *[11, Theorem 3.2.1] Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. Then (X, f) has the periodic shadowing property if and only if $(\mathcal{F}(X), f^{<\omega})$ has the periodic shadowing property.*

Koo and Tsegmid [6] showed that if a dense subdynamical system has the periodic shadowing property, then the original dynamical system also has the same property as below.

LEMMA 2.2. *[6, Theorem 3.1]Let $f : X \to X$ be a homeomorphism and A be an f-invariant dense subspace of a compact metric space X. If $(A, f|_A)$ has the periodic shadowing property, then (X, f) has the periodic shadowing property. Here $f|_A$ is the restriction of f to A.*

LEMMA 2.3. *[11, Theorem 3.2.2] Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. If (X, f) has the periodic shadowing property, then $(2^X, 2^f)$ has the periodic shadowing property.*

Proof. If f has the periodic shadowing property, then $f^{<\omega}$ has the periodic shadowing property by Lemma 2.1. Since $\mathcal{F}(X)$ is a 2^f-invariant dense subset of 2^X, then $(2^X, 2^f)$ has the periodic shadowing property by Lemma 2.2. □

REMARK 2.4. Let $\mathrm{Per}(f)$ denote the set of all periodic points of a homeomorphism $f : X \to X$. If a homeomorphism $f : X \to X$ on a compact metric space X has the periodic shadowing property, then $\overline{\mathrm{Per}(f)} = X$. But the converse does not hold in general.

Proposition 2.5. *If a system (X, f) has a dense set of periodic points, then so does $(2^X, 2^f)$.*

Proof. Let $A \in 2^X$ and $\epsilon > 0$ be given. Then there exists a finite set $\{x_i\}_{i=1}^n$ of X such that $A \subset \bigcup_{i=1}^n B_d(x_i, \frac{\epsilon}{4}) \subset B_d(A, \frac{\epsilon}{2})$. Since $\overline{\mathrm{Per}(f)} = X$, we choose each point $p_i \in \mathrm{Per}(f)$ such that $p_i \in B_d(x_i, \frac{\epsilon}{4})$ for each $i = 1, \cdots, n$. Let $\mathcal{P} := \{p_i\}_{i=1}^n$ be the set of their periodic points of f corresponding to x_i, respectively. Since $p_i \in B_d(x_i, \frac{\epsilon}{4})$ for each $i = 1, \cdots, n$, then $B_d(x_i, \frac{\epsilon}{4}) \subset B_d(p_i, \epsilon)$. We get

$$A \subset \bigcup_{i=1}^{n} B_d(x_i, \frac{\epsilon}{4}) \subset \bigcup_{i=1}^{n} B_d(p_i, \epsilon) = B_d(\mathcal{P}).$$

Similary, we obtain $p_i \in B_d(x_i, \frac{\epsilon}{4}) \subset B_d(A, \frac{\epsilon}{2})$ for each $i = 1, \cdots, n$ and so, $\mathcal{P} \subset B_d(A, \epsilon)$. Thus we have $d_H(A, \mathcal{P}) < \epsilon$, i.e., $\mathcal{P} \in B_{d_H}(A, \epsilon)$. Thus $B_{d_H}(A, \epsilon) \cap \mathrm{Per}(2^f) \neq \emptyset$, i.e., $A \in \overline{\mathrm{Per}(2^f)}$. Hence $\overline{\mathrm{Per}(2^f)} = 2^X$. This completes the proof. $\square$

We give an example to illustrate that the converse of Proposition 2.5 is not true in general.

Example 2.6. *[1, Lemma 1] Let $\prod \mathbb{Z}_2$ be the Cantor space with Abelian topological group structure which group addition on $\prod \mathbb{Z}_2$ is given by*

$$(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (z_1, z_2, z_3, \ldots),$$

where $z_i = x_i + y_i + c_i \pmod 2$, $c_1 = 0$ and $c_i = x_{i-1} + y_{i-1} + c_{i-1}$ (div 2) for $i \geq 2$. Then the group translation homeomorphism $f : \prod \mathbb{Z}_2 \to \prod \mathbb{Z}_2$ defined by

$$f(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) + (1, 0, 0, \ldots)$$

has no periodic points. Also, we see that every cylinder set $[x_1, \ldots, x_n]$ in $\prod \mathbb{Z}_2$ is a periodic point of 2^f in $2^{\prod \mathbb{Z}_2}$ since $f^{2^n}([x_1, \ldots, x_n]) = [x_1, \ldots, x_n]$ for each $n \in \mathbb{N}$. We see that the set of finite unions of cylinder sets forms a dense set of periodic points for 2^f in $2^{\prod \mathbb{Z}_2}$. Hence we see that $\overline{\mathrm{Per}(2^f)} = 2^{\prod \mathbb{Z}_2}$ but $\overline{\mathrm{Per}(f)} \neq \prod \mathbb{Z}_2$.

We obtain the following result that the converse to Lemma 2.3 is true under the expansive condition of the system (X, f).

LEMMA 2.7. *[9, Theorem 4.2.9] Let $f : X \to X$ be an expansive homeomorphism. If $(2^X, 2^f)$ has the periodic shadowing property, then (X, f) has the periodic shadowing property.*

Now, we give a proof of our main result.

Proof of Theorem 1.1. Proof follows from Lemma 2.3 and Lemma 2.7. □

Next, we obtain the following result concerning the strong periodic shadowing property in homeomorphisms and their induced maps.

PROPOSITION 2.8. *Let $f : X \to X$ be an expansive homeomorphism. If $(2^X, 2^f)$ has the strong periodic shadowing property, then (X, f) has the strong periodic shadowing property.*

Proof. Let e be an expansive constant for f. For any $0 < \epsilon < \frac{e}{2}$, there is $\delta > 0$ corresponding to ϵ by the strong periodic shadowing property of 2^f. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a δ-periodic pseudo orbit of f with $x_i = x_{l+i}$ for all $i \in \mathbb{Z}$. Then $\{\{x_n\}\}_{n\in\mathbb{Z}}$ is also a periodic δ-pseudo orbit of 2^f with period l. It follows from the periodic shadowing property of 2^f that there exists $A \in 2^X$ such that $d_H(2^{f^n}(A), \{x_n\}) < \epsilon$ for each $n \in \mathbb{Z}$ and $2^{f^l}(A) = A$ for some $l \in \mathbb{Z}$. Let $a \in A$ be a given point. For each $b \in A$, we have

$$d(f^n(a), f^n(b)) \leq d(f^n(a), x_n) + d(x_n, f^n(b)) \leq \epsilon + \epsilon < e, \ \forall n \in \mathbb{Z}.$$

Then we obtain $A \subset \Gamma_e^f(a)$. Since f is expansive, we have $\Gamma_e^f(a) = \{a\}$, so $A = \{a\}$. Then $d(f^n(a), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. Moreover, $f^l(a) = a$. Hence f has the strong periodic shadowing property. This completes the proof. □

PROPOSITION 2.9. *Let $f : X \to X$ be an expansive homeomorphism and $2^f : 2^X \to 2^X$ be a chain transitive homeomorphism. If a system $(2^X, 2^f)$ has the periodic shadowing property, then the initial system (X, f) has the periodic shadowing property.*

Proof. Suppose that $(2^X, 2^f)$ has the periodic shadowing property. Then the shadowing property of $(2^X, 2^f)$ follows from Theorem 2.5 in [3]. So (X, f) has the shadowing property by [4, Theorem 3.4.]. It follows from Theorem 2.2 in [3] that the shadowing property of f implies the periodic shadowing property. □

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Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: `njkoo@cnu.ac.kr`

**
Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: `avechee@cnu.ac.kr`

***
Department of Mathematics
Mongolian National University of Education
Baga Toiruu, Ulaanbaatar, Mongolia
E-mail: `nyamdavaa.ts@msue.edu.mn`